

Linear Algebra I

28/11/2025, Friday, 18:30 – 20:30

You are NOT allowed to use any type of calculators.

1 Linear equations

2 + 2 + 10 + 3 + 3 = 20 pts

Consider the polynomial $p(x) = a + bx + cx^2$.

- (a) Find linear equations in the unknowns a, b, c by using the conditions

$$p(-1) = p(1) = 2 \quad \text{and} \quad p(0) = 1.$$

- (b) Write down the augmented matrix.

- (c) Put the augmented matrix into the *reduced* row echelon form.

- (d) Determine whether the linear equations are consistent or inconsistent. Justify your answer.

- (e) Find all solutions.

2 Eigenvalues and diagonalization

4 + 7 + 7 + 7 = 25 pts

Let M be the 3×3 matrix given by

$$M = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

where a, b , and c are real numbers.

- (a) By using the relationship between the determinant and eigenvalues of a matrix, show that 0 is an eigenvalue of M .
- (b) By using the definition of eigenvalue, show that $a + b + c$ is an eigenvalue of M .
- (c) By using the relationship between the trace and eigenvalues of a matrix, show that the characteristic polynomial of M is $\lambda^2(\lambda - a - b - c)$.
- (d) Suppose that $a + b + c \neq 0$. Show that M is diagonalizable. Find a diagonalizer.

3 Subspaces of $\mathbb{R}^n$

20 + 5 = 25 pts

Let $a, b \in \mathbb{R}$ and

$$S = \{x \in \mathbb{R}^2 \mid (x_1 - ax_2)(x_1 - bx_2) = 0\}.$$

- (a) Determine all values of a and b such that S is a subspace of $\mathbb{R}^2$.
- (b) Find a basis for S whenever it is a subspace.

4 Cayley-Hamilton theorem

5 + 15 = 20 pts

Let $M \in \mathbb{R}^{3 \times 3}$ be such that $\det(M) = 1$ and $\frac{-1 + \sqrt{3}i}{2}$ is an eigenvalue of M .

- (a) Find all eigenvalues of M .

- (b) Let $M^{100} = aM^2 + bM + cI_3$. Using Cayley-Hamilton theorem, determine a, b, c .

10 pts free